On Passive Damping Mechanisms in Large Space Structures

Holt Ashley*
Stanford University, Stanford, California

The significance of even tiny amounts of passive energy dissipation to ensure successful stabilization of large, flexible space structures is explained. The study of scale effects on various mechanisms indicates that modal damping ratios are likely to decrease as size increases in a family of similar structures. This paper focuses on thermal dissipation induced by strain gradients during vibration of monolithic configurations. Past work and the expected magnitudes of this damping are reviewed, along with reasons why it is, to some degree, under the designer's control. In the search for the highest practical values, unidirectional metallic composites and other arrangements are examined.

Nomenclature

$a_n(\tau)$ = time dependence of nth mode, Fourier coefficient
b = depth of rectangular beam
c_v = constant-strain specific heat (unit mass)
E = Young's modulus
f = friction force in joint
i = imaginary unit, $\sqrt{-1}$
$\mathbf{i}, \mathbf{j}, \mathbf{k}$ = Cartesian unit vectors
I = area moment of inertia of beam section
k = thermal conductivity
L = reference length, length of bar or beam
m = modal index integer
M = bending moment
N = compressive force normal to joint
$\mathbf{q}$ = displacement vector
s = specific entropy, Laplace transform variable
$\mathbf{S}$ = entropy displacement vector
t = time coordinate
t_0 = (small) dimensionless material constant, $= T_0 E\alpha^2/\rho c_v$
T = absolute temperature (ΔT is perturbation $T - T_0$)
T_0 = reference temperature
u, v, w = Cartesian components of $\mathbf{q}$
U = spatial amplitude of u
x, y = coordinates parallel and normal to length of beam or bar ($\bar{y} = y/b$)
α = coefficient of linear thermal expansion
β = complex wavenumber
γ = dimensionless parameter in Eqs. (18) and (19)
δ = (small) dimensionless constant, $= k/Lc_v\sqrt{\rho E}$
Δ = material constant [Eq. (7)], increment
ϵ = engineering strain (with subscripts)
ζ = critical damping ratio of oscillation
η = loss factor
θ = dimensionless constant ($\cong 1$)
λ = dimensionless frequency, $= \omega L/\sqrt{\rho/E}$
μ = friction coefficient
ν = Poisson's ratio
ξ = dimensionless coordinate along bar
ρ = density
σ = stress (with subscripts)
τ = dimensionless time, time constant in Eq. (7)
ω = circular frequency
Ω = spacecraft angular velocity, dimensionless parameter in Eq. (19)
∇ = vector Nabla operator
$(\cdots)_i$ = property of i^{th} component of composite material
$(\cdots)_n$ = property of nth vibration mode

Introduction

THE design, construction, and control of large space structures (LSS) lately have become popular foci of aerospace technological research. They are fields that one hopes will progress beyond the laboratory and the drawing board before the end of the century. Numerous surveys, typified by Bekey and Naugle,[1] describe the configurations and dimensions of candidate systems that will rely on LSS for their successful performance. Linear scales in the kilometer range are mentioned. Requirements on control systems for telescopes and antennas refer to pointing accuracies in the hundredths of arc seconds or less, as well as to holding the figure of 100-m-diam or larger reflectors to a precision of one-tenth of a 1-cm wavelength. It is not difficult to make the case that even very small amounts of passive energy dissipation are essential for the practical satisfaction of such criteria.

In the extensive literature on active control of LSS, the role of structural damping is frequently omitted or relegated to the inclusion of little first-derivative terms—often of unspecified numerical magnitude—in the equations of motion. For example, the excellent study on controllability and observability by Hughes and Skelton[2] neglects it entirely, stating "the addition of slight damping... is not of practical significance" for the questions considered there. Quite true, but it is also known that satisfactory performance and sometimes stability of active systems with finite bandwidth rely essentially on the presence of such damping.

More than 10 years ago Gevarter[3] made a fundamental study touching this issue. He concluded, "For vehicles which can be characterized as beams, it is unlikely that all flexible modes can be stabilized without structural damping," except by arrangements of sensors and effectors which seem quite impractical for large space systems. Cannon[4] recently pointed out to the author that, unless the concept of colocated sensors and controls (cf., Wykes[5]) is fully implemented, it is impossi-

Presented as Paper 82-0639 at the AIAA/ASME/ASLE/AHS 23rd Structures, Structural Dynamics and Materials Conference, New Orleans, La., May 10-12, 1982; submitted May 24, 1982; revision received Aug. 2, 1983. Copyright © American Institute of Aeronautics and Astronautics, Inc., 1982. All rights reserved.

*Professor, Aeronautics/Astronautics and Mechanical Engineering. Honorary Fellow AIAA.

ble to stabilize normal vibration modes whose frequencies exceed some limit determined by the feedback transfer functions. Since these facts are now generally recognized, one is not surprised to see revived interest in the very old subject of material damping and its applications. For LSS, moreover, the needs are uniquely important. Unlike large vehicles in the atmosphere, spacecraft are not surrounded by an environment which always provides a potential external source of energy dissipation. The lowest natural frequencies may be one to three powers of ten less than for typical aircraft vibrations. In cases of plate- or shell-like structures their spectra will be "mode rich." Their control system designers may, furthermore, have to account for several decades of these spectra. These and other considerations point to requirements for prediction methods and accurate information about damping.

Among the several known mechanisms of internal dissipation three are usually distinguished and considered seriously in connection with LSS (cf., surveys in Refs. 6-8, out of many publications that could be cited):

1) Damping inherent to the material under dynamic excitation.

2) Damping at interconnections, joints, bearings, etc. (often modeled as Coulomb friction; see, for example, the survey by Plunkett[9]).

3) Damping furnished artificially by dashpots (Horner[10]) or by more practicable means such as constrained viscoelastic layers (Refs. 11 and 12 are recent studies).

One or more of these mechanisms is incorporated in each of several finite element programs that are being adapted for LSS. Ottens[13] tabulates the linear approximations embodied in NASTRAN and 14 other general-purpose codes. The 1981 paper by Bogner and Soni[14] (see also Brockman[15]) describes a program MAGNA which, at the time, idealized joint friction with an equivalent concentrated linearized force. Reference 14 also includes material damping in terms of the complex "loss factor," about which more is said subsequently.

Now artificial passive devices are the most powerful means of dissipation available to the designer, but their inevitable weight penalty may not be acceptable in LSS, where they would have to be widely dispersed throughout the structure. They are, therefore, omitted from the present investigation. Except in the section on scale effects, only the first listed mechanism is studied. Only small amplitudes of free and forced motion are treated, because they are believed to have special significance for LSS and because nonlinearity invariably seems to produce larger relative damping as the amplitude of strain increases (cf., Crandall,[16] surveys such as Refs. 6, 7, 17, and Sec. 82 of Ref. 18). The aim here is to seek lower bounds on the phenomenon and assess the degree to which these bounds may be affected by the choice of material and configuration. It is in metallic-alloy structures where these lower bounds are expected to occur, therefore, the paper concentrates on metals. Included is a comparison between the classical theory and recent damping measurements in a simulated space environment.

Effects of Scale in LSS

As an estimate of what might be anticipated when the size range forecast for LSS is attained in practice, the author has examined in some detail the effects on modal critical damping ratios ζ_n which would be produced by changing a typical length dimension L in a family of geometrically similar structures of the same materials. Linear or equivalent linear behavior is assumed. Space limitations forbid a full presentation of the work here, but some key results will be reproduced.

As a preliminary step, it is not unreasonable to omit contributions from joints and study the purely "monolithic" case. Dimensional analysis serves the purpose of illustration, but comparable conclusions are found from the full structural-dynamic treatment of many special cases. Let it be supposed that the ζ_n and corresponding frequencies ω_n are functions of L; density ρ; typical elastic properties† like Young's modulus E and Poisson's ratio ν; and a single measure of energy dissipation. For the last, the loss factor or loss coefficient η (see Ref. 18, p. 34 and Fig. 2.6) is convenient. In nearly simple harmonic motion it is defined by replacing E with the complex constant

$$E = E_R[1 + i\eta] \tag{1}$$

As usual, vibratory quantities are represented by real parts of expressions containing the factor $e^{i\omega t}$. It is easily shown that

$$\omega_n = \frac{1}{L}\sqrt{\frac{E}{\rho}}\, f(\nu, \eta) \tag{2}$$

$$\zeta_n = g\left(\frac{\omega_n L}{\sqrt{E/\rho}}, \nu, \eta(\omega_n)\right) \tag{3}$$

where f and g are dimensionless functions that must be determined from test or physical theory. Because of the smallness of the dimensionless parameters ζ and η, it is likely that η appears as a linear factor in Eq. (3). These parameters are not, however, independent of frequency. As discussed below in connection with Eqs. (7), (22), etc., the lower modes of metallic LSS are likely to occur in a range where $\eta \sim \omega_n$, approximately. Since the ω_n decrease with increasing L, damping will do likewise.

The presence of joints is accounted for next in the LSS. The area of each joint must vary in proportion to L^2, whereas the volumes and masses of monolithic elements grow with L^3 as size is increased. Let it be assumed that the behavior of joint friction lies somewhere between linear viscous damping and Coulomb's law, the latter implying a resistance proportional to the normal or clamping force across the joint (see below). For the former purely viscous mechanism of dissipation, the foregoing "square-cube" considerations lead without difficulty to

$$\zeta_n \sim L^{-1} \tag{4}$$

The limit of solid friction can be studied by somehow linearizing the relation (cf., Bogner and Soni[14]) for the force opposing the relative tangential velocity $\dot{x}_k$ at the kth joint:

$$f_k = \mu_k N_k \operatorname{sign}(\dot{x}_k) \tag{5}$$

Here μ_k is a constant friction coefficient. To estimate the influence of scale L, one must hypothesize the dependence of N_k on L. (This is complicated by the problem of "breakout" —if N_k is too large or the motion amplitude too small, dissipation may almost disappear because no relative motion takes place.) In general N_k is determined from some combination of the local clamping at the joint and the normal component of the equilibrium force system present within the structure. For purposes of argument, let one hypothesize that the latter dominates and that it is a compressive centrifugal force due to rotation of the structure at angular velocity Ω. In this special case, one sees that N_k is proportional to $\Omega\rho L^4$ in the family of similar LSS. It can then be reasoned that the equivalent linear dissipation gives rise to (small)

$$\zeta_n \sim (\Omega L/v_c)^2 \tag{6}$$

where $v_c \equiv \sqrt{E/\rho}$ is a p wavespeed characteristic of the material. Centrifugal stresses are proportional to $\rho\Omega L^2$; this and other facts suggest that the factor ΩL will probably

†Isotropy is implied by the use of two constants; nonisotropic materials are covered, however, because their other linear elastic constants occur in fixed proportions to one another.

decrease with increasing L. Once again the indication is that ζ_n will go down as the LSS scale increases.

All the evidence, therefore, points to smaller damping ratios for larger structures in space. With two or more damping mechanisms at work, a dependence of ζ_n somewhere between the zeroth and -1 power of L is projected. Clearly the subject of lower bounds at small amplitudes deserves further study.

Available Sources of Material Damping

The remainder of the paper concentrates on LSS constructed of crystalline solid members, which include, of course, the alloys of aluminum, titanium, etc. Advanced reinforced plastics are omitted, not because they are unattractive candidate materials, but because their energy dissipation is less susceptible to simple analysis (cf., however, surveys and data sources like those discussed by Bert[19]). At the small strain amplitudes of interest here, crystal damping can be explained in terms of the phenomenon known as "anelastic relaxation," wherein the full amount of strain due to a given application of stress requires a finite time for its development.

The text by Nowick and Berry,[20] a valuable source of theory and data on the subject, will be cited repeatedly. The first six chapters of Ref. 20 describe the formal theory; for instance, Chap. 3 provides a full treatment of the familiar Voigt model and the "standard anelastic solid." A particularly useful concept that approximates many actual situations is the Debye formula, according to which some measure of (small) oscillatory damping such as ζ or η is predicted to be proportional to

$$F(\omega) \cong \Delta \frac{\omega\tau}{1+\omega^2\tau^2} \tag{7}$$

Here Δ is a temperature-dependent material constant, and τ is characteristic time that can be controlled by the choice of material, specimen shape and size. Peak damping obviously occurs when frequency $\omega = \tau^{-1}$, so that an opportunity exists for making the spectral range of this maximum a matter of design. (It is worth noting that many complex solids display several such "Debye peaks" or even a continuous distribution.)

Most of the research summarized in books such as Refs. 18 and 20 was conducted by crystallographers or metallurgists. Through a combination of theory and laboratory tests, it had as goals the measurement of certain basic properties of the crystals or the underlying atomic/molecular structure. Those interested in damping per se are therefore the unintended beneficiaries of a vast literature. One product of all of this work was the identification of many different relaxation mechanisms, occurring at various scales from lattice size on up. It is instructive to examine which of these could be employed in aerospace structures.

Table 1 lists and comments on six such mechanisms. In order to avoid an excessively long bibliography, references are made to the sections in Ref. 20 where the particular topic is discussed with citations. Descriptions of the phenomena are necessarily telegraphic; of more significance are the hypothesized advantages and disadvantages relative to LSS applications. It is believed that most of the phenomena in Table 1 have possible utility for structural damping in practice. The metals in which most of them were investigated, however, differ markedly from the well-characterized, lightweight, and high-strength alloys normally chosen for spacecraft. Accordingly—and with the proviso that mechanisms like the Snoek, Zener, and grain-boundary relaxations deserve study for LSS —thermoelastic damping is selected for the purposes of this paper. It is almost universal, in the sense that dynamic straining alters the temperature in any substance with nonzero thermal expansivity. It is present, as a linear effect, down to the smallest vibration amplitudes. It leaves the structure unaltered, because after a transient everything is able to settle back to its initial state except for a small increase in the entropy of the surroundings. Within limits, it can be designed into the material or the structural arrangement of a LSS, and the same members that dissipate energy will also carry the loads and furnish the stiffness.

Theory and Simple Examples of Thermoelastic Damping

Zener[22-24] was apparently the first to explain clearly how thermoelasticity gives rise to vibrational damping. His predictions received quick and accurate confirmation, notably by

Table 1 Typical relaxation phenomena which might contribute to LSS damping

Name	Mechanism	Features favorable to applications	Features unfavorable to applications
Magnetoelastic relaxation (Ref. 20, Secs. 18.1-18.5)	Eddy currents due to stress-induced changes of magnetization.	High damping possible at low temperatures. Peaks *can* occur at interesting frequencies.	Ferromagnetic materials only. Depends on degree of magnetization.
Snoek and other point-defect relaxations (Ref. 20, Secs. 7.1-9.3)	Movement of point defects due to straining (e.g., Snoek involves interstitial solute in bcc crystal).	Moderate peaks can occur at low temperatures and interesting frequencies.	Only seen in iron and other body-centered-cubic metals, with H, C, N, or O in *dilute* solution. Damping very composition-dependent.
Zener relaxation (Ref. 20, Secs. 10.1-10.5)	Stress-induced reorientation of pairs of solute atoms.	η greater than 10% has been observed in particular alloys. Peaks *can* occur at interesting frequencies and temperatures.	Best alloys studied at low temperature have little aerospace interest (see Ref. 20, Table 10-1); η depends on crystal orientation.
Dislocational relaxations (several mechanisms; Ref. 20, Secs. 13.1-14.4)	Different mechanisms, involving propagation of dislocations.	Seen in many different metals from pure single crystals to alloys. Moderate peaks at various temperatures.	Peaks tend to occur at very high frequencies. Material degradation may occur. Alloys seem to have little aerospace interest.
Grain-boundary relaxation (Ref. 20, Secs. 15.1-15.6)	Viscous slipping at boundaries between grains.	Can occur in many polycrystalline systems, but very composition-dependent. η values approaching 10% have been observed.	Most data available on nearly pure metals. Tends to peak at temperatures much higher than expected in LSS.
"Large-scale" thermoelastic relaxation (Ref. 20, Secs. 17.1-17.5; Ref. 21, Chap. 5, and other texts)	Heat flux due to time-dependent strain gradients.	Observed in all materials. Low to moderate peaks can occur at low temperatures and interesting frequencies. Capable of accurate theoretical prediction.	Magnitude and location of peaks highly dependent on configuration, material system, and temperature.

some tests of Bennewitz and Rötger[25] on small cantilever beams made of German silver. The literature on the subject is enormous, but the author found the treatments by Biot,[26] Nowick and Berry[20] (Chap. 17), and Lazan[18] (pp. 42-43 and elsewhere) especially useful. Besides the nearly isotropic metals, analyses have been published for anisotropic materials, the work of Tasi and Herrmann[27,28] on crystal plates being a good example.

Thermoelasticity involves a small interaction between elastomechanical and thermodynamic behavior in a solid. Accompanying his derivations with an illuminating physical discussion, Biot[26] observes that the former are describable in terms of either the tensor of stress or strain. These are conjugate variables because the inner product of one by a variation in the other yields work or energy change. Similarly, the natural thermodynamic conjugates are absolute temperature T and specific entropy s. In a fashion similar to the connection between strain and the displacement vector

$$q = ui + vj + wk \tag{8}$$

Ref. 26 introduces an "entropy displacement vector" S defined by

$$s = \nabla \cdot S \tag{9}$$

For present purposes T and strain (actually q), as functions of position and time t, are the quantities used.

Following Ref. 26 or Secs. 2-2 and 2-3 of Ref. 29, one writes the six strain-temperature-stress relations for homogeneous, isotropic material. These are typified in Cartesian coordinates by the x component of extensional strain:

$$\epsilon_{xx} \equiv \frac{\partial u}{\partial x} = \alpha \Delta T + \frac{\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})}{E} \tag{10}$$

The common definitions are adopted here, with α the coefficient of linear expansion and $\Delta(\cdots)$ denoting the perturbation in a quantity. At the level of small displacements the equations of dynamic equilibrium and compatibility are unaffected by temperature perturbations.

The other key relation in the coupling is obtained by combining the first and second laws of thermodynamics with the Fourier law of heat conduction. It can be written as follows:

$$k\nabla^2(\Delta T) - \rho c_v \frac{\partial(\Delta T)}{\Delta t} - \frac{T\alpha E}{(1-2\nu)} \nabla \cdot \frac{\partial q}{\partial t} = 0 \tag{11}$$

Here k is thermal conductivity and c_v a constant-volume specific heat referred to unit mass. Of course k, α, and E must be replaced by tensors for anisotropic materials (cf. Tasi[27]).

Boundary and initial conditions need to be furnished for Eqs. (10) and (11) and the dynamic equilibrium equations. The former include prescribed stress and/or displacement, and a single condition must be enforced on T at the bounding surfaces. In outer space at low skin temperatures this is the insulated-surface requirement of zero normal temperature gradient; for hotter skins and very low frequencies one might have to account for heat loss due to radiation.

Very few "exact solutions" for thermoelastic damping are known to the author.‡ A great deal can be learned, however, from such approximations as simple harmonic motion and/or assumed modes for the spatial temperature variation. As an illustration which permits a useful tabulation of loss-factor values for various metals, consider Zener's problem[22,23]: bending vibration of a beam with rectangular cross section or a plate of constant thickness.

One solution for the homogeneous, isotropic beam will be summarized. As far as "monolithic" damping is concerned, the results do not differ significantly between beam and plate or under different mechanical boundary conditions.

With x and y longitudinal and normal coordinates, respectively, in the plane of bending, let it be assumed that ΔT, the bending stress σ_{xx}, and the strain ϵ_{xx} vary rapidly only with y. Under the Bernoulli-Euler approximation and for sinusoidal oscillating of frequency ω, one can write

$$\epsilon_{xx} = -e^{i\omega t}\frac{M_0 y}{EI}F(\xi) \tag{12}$$

Here M_0 is a reference bending moment. The function $F(\xi)$, with $\xi = x/L$, is a mode of flexural curvature which is determined by the manner of excitation and the mechanical boundary conditions. For instance, if the beam were simply supported but driven at the end $\xi = 1$ by a concentrated moment $M_0 e^{i\omega t}$, it is easily shown that§

$$F(\xi) = \frac{1}{2}\left[\frac{\sin(\sqrt{\lambda}\,\xi)}{\sin\sqrt{\lambda}} + \frac{\sinh(\sqrt{\lambda}\,\xi)}{\sinh\sqrt{\lambda}}\right] \tag{13}$$

The behavior of temperature perturbation $\Delta T(y,t) \ll T_0$ is determined from Eq. (11). Under conditions of uniaxial stress the volumetric strain is given by

$$\nabla \cdot q \cong \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} = (1-2\nu)\epsilon_{xx} + 2(1-\nu)\alpha\Delta T \tag{14}$$

where the relations like Eq. (10) have been used to introduce temperature effects. The approximations (12-14) are substituted into Eq. (11), $\partial/\partial t$ is replaced by $i\omega$, and a dimensionless coordinate $\bar{y} = y/b$ is introduced. With $T \cong T_0$, the resulting differential equation for the temperature perturbation can be written as

$$\frac{\mathrm{d}^2(\alpha\Delta T)}{\mathrm{d}\bar{y}^2} - \frac{i\omega b^2 \rho c_v}{k}\theta(\alpha\Delta T) = -i\omega\bar{y}\frac{M_0 b}{I}\left(\frac{T_0\alpha^2 b^2}{k}\right)F(\xi) \tag{15}$$

Since $e^{i\omega t}$ has been cancelled, $\Delta T(\bar{y})$ now stands for the complex amplitude of temperature variation. The parameter $\theta \cong 1$ is connected with the ratio of specific heats at constant stress and strain:

$$\theta = 1 + 2\left(\frac{1+\nu}{1-2\nu}\right)\frac{T_0 E\alpha^2}{\rho c_v} \equiv 1 + 2\left(\frac{1+\nu}{1-2\nu}\right)t_0 \tag{16}$$

where t_0 is a small but important parameter which couples temperature with the material properties.

Zener's approach to solving Eq. (15) was to replace $\Delta T(\bar{y})$ with a Fourier series, each of whose terms satisfies the thermal boundary conditions at top and bottom faces. For example, $\mathrm{d}(\alpha\Delta T)/\mathrm{d}\bar{y}$ must vanish at insulated surfaces $\bar{y} = \pm\frac{1}{2}$, which suggests,

$$\alpha\Delta T = \sum_{n=0} a_n \sin(2n+1)\pi\bar{y} \tag{17}$$

For practical materials, predictions of three-figure accuracy were achieved by retaining only the first term in Eq. (17). It is of possible interest that essentially the same results are found

‡See, however, extensive work on wave propagation as summarized in Chaps. 20 and 21 of Ref. 30.

§$\lambda \equiv \omega L^2\sqrt{m/EI}$ in Eq. (13) only.

by directly solving Eq. (15) for insulated boundaries:

$$\alpha \Delta T(\eta) = \frac{t_0}{\theta}\left(\frac{M_0 b}{EI}\right) F(\xi)\left[\bar{y} - \frac{\sinh\gamma\bar{y}}{\gamma\cosh\frac{\gamma}{2}}\right] \tag{18}$$

In the homogeneous part of Eq. (18) there appears a complex parameter

$$\gamma = (1+i)\sqrt{\frac{\theta}{2}\left(\frac{\omega b^2 \rho c_v}{k}\right)} \equiv (1+i)\sqrt{\theta\frac{\Omega}{2}} \tag{19}$$

From the definition of Eq. (1) of loss factor η one can derive an averaged measure of damping in the beam.

$$\overline{\frac{E_R}{E}[1+i\eta]} = \overline{\left(\frac{\sigma_{xx}}{E\epsilon_{xx}}\right)} = 1 - \overline{\left(\frac{\alpha\Delta T}{\epsilon_{xx}}\right)}$$

$$= 1 + \frac{t_0}{\theta}\overline{\left[1 - \frac{\sinh\gamma\bar{y}}{\gamma\bar{y}\cosh\frac{\gamma}{2}}\right]} \tag{20}$$

Here Eqs. (12) and (18) have been substituted; the resulting cancellation of $F(\xi)$ and constants such as M_0 related to the mechanical response verify an earlier statement regarding η's insensitivity to these details.

The overbar in Eq. (20) calls for taking the mean, which simply involves integrating through the depth $-\frac{1}{2} \leq \bar{y} \leq \frac{1}{2}$. Since $(E_R/E) \cong 1$ closely and t_0/θ is real, one obtains

$$\bar{\eta} \cong \text{Im}\left\{\frac{-2(t_0/\theta)}{\gamma\cosh(\gamma/2)}\int_0^{\frac{1}{2}}\frac{\sinh\gamma\bar{y}}{\bar{y}}\,\mathrm{d}\bar{y}\right\} \tag{21}$$

Although the integral here is a tabulated function known as Shi$(\gamma/2)$, it is much easier to take the imaginary part of all complex quantities first and then use Simpson's quadrature rule for numerical evaluations. For a given material and beam depth b, $\bar{\eta}$ depends only on frequency ω. Within a percent or so, its values are found to agree with the Debye formula [Eq. (7)]. The two principal parameters in Eq. (7) are as follows:

$$\tau = \left(\frac{b}{\pi}\right)^2 \frac{\rho c_v}{k}\theta \tag{22a}$$

$$\Delta = \frac{10.97}{\pi^2}\frac{t_0}{\theta} = 1.111\left[\frac{T_0 E\alpha^2/\rho c_v}{1 + 2\left(\frac{1+\nu}{1-2\nu}\right)\frac{T_0 E\alpha^2}{\rho c_v}}\right] \tag{22b}$$

As summarized in Fig. 2.6 of Ref. 18, the factor $\bar{\eta}$, the so-called "quality" Q, and the damping ratio of a corresponding free vibration are (when $\bar{\eta} \ll 1$) connected by

$$\zeta = \frac{\bar{\eta}}{2} = \frac{1}{2Q} \tag{23}$$

Since ζ is used here, it can be estimated by halving the Debye values from Eq. (7). The maximum of ζ is thus $\Delta/4$ and occurs when $\omega = \tau^{-1}$.

Some idea of the possibilities of flexural thermoelastic damping can be inferred from Table 2. Here estimates of Δ and τ^{-1} are listed for several metals and their alloys. Properties required for the calculations were taken from sources such as Eshbach and Souders[31] and Reynolds and Perkins.[32] The table shows that aluminum holds the greatest promise among the common aerospace alloys, due mainly to its high thermal expansivity.

When interpreting Table 2 it is important to distinguish between the levels of available damping and the frequency range where they peak. The factor Δ is determined by the material but is also proportional to T_0, which suggests that favorable effects might be obtained by adjusting the radiation balance of LSS to keep temperature high. Δ is, however, independent of ω and the structural scale. On the other hand, ω for peak damping is inversely proportional to depth (or thickness) squared. Quantities are tabulated for $b = 0.1$ m. But Eq. (22a) shows that $\omega_{max} = 0.0831$ rad/s (75.5-s period) for aluminum could be increased to 8.31 rad/s just by reducing b to 1 cm. There is an interesting consistency between this size dependence and the fact that smaller structures tend to have higher natural frequencies. Finally, one remarks that the "half-power" points on the curve of ζ vs ω occur at

$$\omega = (\sqrt{2} \pm 1)\,\omega_{max} \tag{24}$$

A higher ω_{max} is, therefore, associated with a broader peak in terms of absolute frequency range.

Table 2 Constants of the Debye peak for flexural damping of metallic beams[a]

Metal	Δ Eq. (22) ($T_0 = 300$ K), $\times 10^{-3}$	$\omega_{max} = \tau^{-1}$, rad/s ($b = 0.1$ m)
Al and alloys	6.10	0.0831
Cu	2.91	0.1121
Low-C steel	2.70	0.0225
Ti and alloys	0.72	0.0075
Ni and alloys	3.16	0.0141
Mg	5.37	0.0844

[a]It has been found for plates that values of Δ are greater by a factor $2/(1-\nu)$.

Even under the best laboratory circumstances the precise measurement of free or forced energy dissipation is very demanding.¶ In configurations that might simulate scaled LSS, accuracy can be spoiled by such effects as "air damping" and nonrepresentative boundary conditions. A recent opportunity for comparison between theory and tests on aluminum beams is offered by the work of Vorlicek[33] at M.I.T., who measured the decay of natural vibrations of beams projected upward into free flight, for short intervals, in a vacuum chamber. The frequency of the fundamental free-free mode was adjusted by cutting various lengths. Figure 1 gives the Ref. 33 data and predictions by Eq. (7) and Table 2, plotted on linear scales (unlike Fig. 5.8 of the reference) to emphasize differences. The agreement is certainly not unsatisfactory in view of experimental difficulties. Except at the highest value of ω, theory seems to furnish a lower bound for the small-amplitude measurements of damping ratio. Incidentally, Vorlicek[33] also tested two graphite/epoxy composite beams. The desirability of this material for LSS is emphasized by the data, which show ζ in a range between about 7 and 20×10^{-3}, highest values were found for the case of uniaxial reinforcement parallel to the long axis.

The significance for damping of how the members in a structure carry their dynamic loads can be seen by contrasting the foregoing ζ estimates on beams and plates with other familiar configurations. The opposite extreme consists of a member subjected to pure shear strain, such as a monolithic rod of solid or hollow circular section. If it is also isotropic, the volumetric strain $\nabla \cdot \boldsymbol{q}$ in Eq. (11) vanishes identically. Accordingly, no thermoelastic dissipation is expected at all.

Another case which *appears* equally simple involves a bar of uniform sectional area, undergoing longitudinal vibration. An elementary physical discussion of the bar, explaining why only infinitesimal damping can be expected with peaks at extreme frequencies on the order of 3×10^{10} rad/s, is pre-

¶Chapter 20 of Ref. 20 gives a useful discussion of experimental methods.

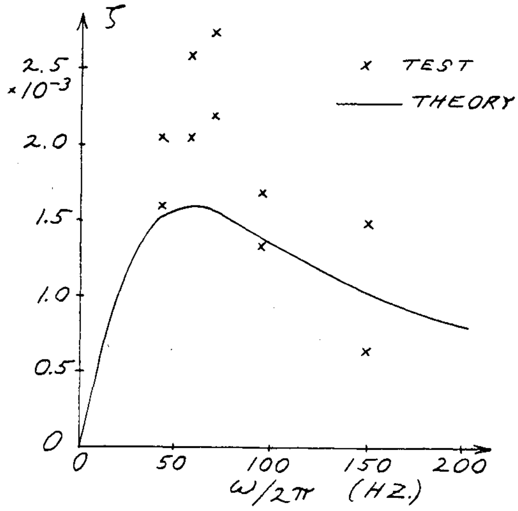

Fig. 1 Comparison between the Zener peak for aluminum and measured damping ratios of free-free beams.[33] Beam depth $b = 0.1575$ cm.

sented in Ref. 20, Sec. 17.4. There should, one thinks, be some influence of boundary conditions and other details on the temperature and strain distributions. More comprehensive forced and transient analyses of the bar have, therefore, been conducted just for their intrinsic interest.

Without setting down mathematical details, the author can state that he has obtained a directly time-dependent transient solution for thermoelastic effects on bar vibration. After Laplace transformation to the s plane, one discovers that free motion in the bar's mth normal mode is governed by a cubic characteristic equation:

$$s^3+(m\pi)^2\frac{\delta}{\theta}s^2+(m\pi)^2\left[1+\frac{t_0}{\theta}\right]s+(m\pi)^4\frac{\delta}{\theta}=0 \quad (25)$$

Here θ and t_0 are given by Eq. (16). δ is a new parameter defined by

$$\delta=\frac{k}{Lc_v\sqrt{\rho E}} \ll 1 \quad (26)$$

For a small bar with $L = 1$ cm in aluminum, typical calculations based on Eq. (25) yield a ζ peak near 4×10^{-3}, with m_{max} a small multiple of 10^5. As L increases, the frequency at which this peak occurs increases further, since m_{max} grows in proportion to L^2. Thus the statement of Ref. 20 is confirmed: the metallic bar is indeed an uninteresting example!

The Design of LSS for Internal Damping

It is believed that the preceding section serves to support the author's claim that principles can be developed whereby designers of LSS can try to ensure that desirable amounts of passive damping are present in the primary structure. Further investigations should first concentrate on small amplitudes of free and forced response. Relative damping is almost certain to increase with amplitude[16] so that the associated nonlinearity will enhance whatever open- or closed-loop stability is available in the smaller motions. The extensive use of inserts which dissipate energy without contributing to strength and stiffness seems unattractive, except as a last resort, because of the weight penalty.

Another topic that deserves further investigation, beyond the extensive literature on reinforced plastics, concerns the potential for augmenting thermoelastic damping by means of the other "composites" made of dissimilar materials. A metallic example would be boron-reinforced aluminum. The question that arises is whether greater dissipative temperature gradients

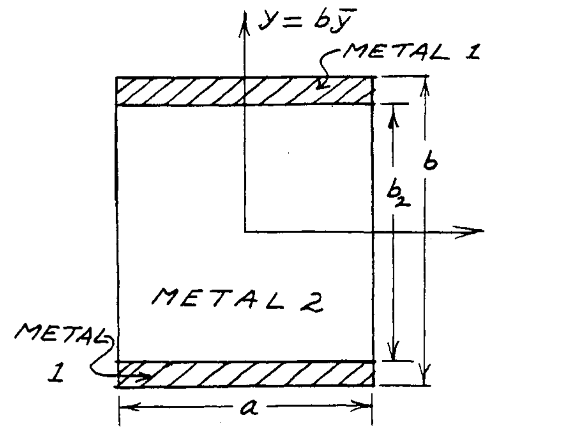

Fig. 2 Idealized bimetallic or "composite" beam cross section. *x* is normal to page, with bending in *x-y* plane. (Analysis is unaffected by width *a*.)

can be generated through the requirements for strain, temperature, and normal-heat-flux compatibility at interfaces. Naturally one looks first at flexural members.

As the simplest possible illustration, Zener's analysis[22,23] has been generalized for a bimetallic beam of rectangular cross section (Fig. 2). The picture suggests a core of parent material (properties identified by subscript *2*), reinforced at its outer faces by rectangular slabs [properties $(\cdots)_1$] which occupy a fraction $(1-b_2/b)$ of the depth. It might conceivably idealize some outer layers of, e.g., unidirectional boron-fiber reinforcement with the boron "smeared" into two equivalent rectangular beam caps.

By analogy with Eq. (12), assume simple harmonic motion and a linear distribution of axial strain

$$\epsilon_{xx}=-\bar{y}G(\xi)e^{i\omega t} \quad (27)$$

where $G(\xi)$ is a curvature distribution determined from the boundary conditions and manner of excitation. Following the analysis of the single-material beam, one writes dimensionless differential equations for the temperatures in the two layers:

$$\frac{d^2(\alpha_j\Delta T_j)}{d\bar{y}^2}-i\theta_j\Omega_j(\alpha_j\Delta T_j)=-i\bar{y}\Omega_j t_{0j}G(\xi) \qquad \text{for } j=1,2 \quad (28)$$

ΔT must be continuous across the interfaces at $\bar{y}=\pm\bar{y}_2=\pm b_2/2b$. There are also boundary conditions of zero and continuous heat flux, respectively, at $\bar{y}=\pm\frac{1}{2}$ and $\pm\bar{y}_2$:

$$\frac{d\Delta T_1(\pm\frac{1}{2})}{d\bar{y}}=0 \quad (29)$$

$$k_1\frac{d\Delta T_1(\pm\bar{y}_2)}{d\bar{y}}=k_2\frac{d\Delta T_2(\pm\bar{y}_2)}{d\bar{y}} \quad (30)$$

The parameter Ω_j is

$$\Omega_j=\frac{\omega b^2\rho_j c_{vj}}{k_j} \quad (31)$$

Although a solution analogous to Eqs. (18) and (19) is feasible, it is easier to work with Fourier series such as Eq. (17). In fact, one can substitute Eq. (17) directly into Eq. (28) for the respective regions, truncate the series, and construct simultaneous equations for the constants a_n by multiplication with individual sine terms and integration through the depth. The resulting series for $\alpha\Delta T(\bar{y})$ appears, as a rule, to converge so rapidly that only one or two terms need be retained. In the bimetallic case, however, careful attention is thereby sacrificed to heat-flux condition (30), unless the conductivities

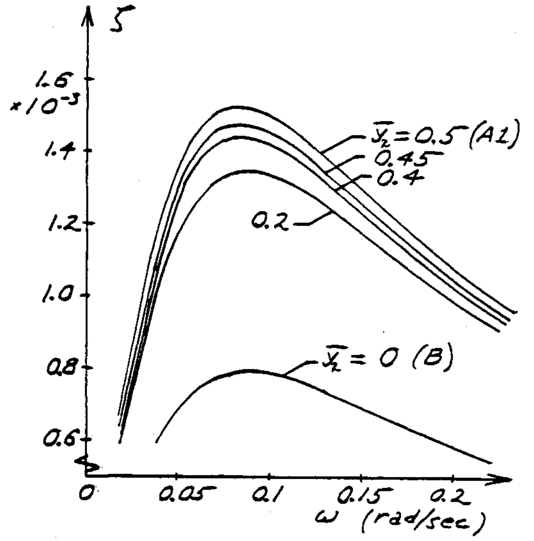

Fig. 3 Damping ratios predicted in the vicinity of the Debye peak for bending vibration of the boron/aluminum "composite" beam shown in Fig. 2. The depth $b = 10$ cm; parameter on the curves is proportional to the core-height fraction $2\bar{y}_2$.

Table 3 Influence of expansivity ratio α_2/α_1 and core-height fraction $2\bar{y}_2$ on the maximum damping ratio attainable by a bimetallic "composite" beam. $T_0 = 300$ K; depth $b_1 = 10$ cm ($\alpha_2/\alpha_1 = 5.75$ corresponds to Al and tungsten)

α_2/α_1	$\bar{y}_2$	Approx ω_{max}, rad/s	Maxζ, $\times 10^{-3}$
5.75	0.4	0.08	1.52
5.75	0.3	0.08	1.63
5.75	0.2	0.08	1.78
5.75	0.1	0.08	1.55
5.75	0.05	0.08	1.20
20	0.4	0.08	1.59
20	0.3	0.08	1.91
20	0.2	0.08	2.73
20	0.1	0.08	3.35
20	0.05	0.08	2.41
100	0.4	0.08	1.61
100	0.2	0.08	3.43
100	0.05	0.088	8.29
200	0.05	0.09	18.2
500	0.05	0.09	28.7
1000	0.05	0.09	33.1

k_1 and k_2 are nearly equal. (Fortunately this may be true for B/Al; k for Al is 217.5 N/s-K, whereas for boron it is estimated** to be 270 N/s-K.)

After quite a bit of algebraic manipulation, one can derive an expression for the averaged loss factor, which generalizes steps like those in Eqs. (20) and (21). When $\overline{E_R/E}$ is approximated by unity and only one term is retained in series (17), the result can be written as follows:

$$\bar{\eta} = 2\zeta \cong \frac{8}{\pi^2} t_{02}\left(\frac{\omega\tau_2}{\theta_2}\right)\left[1.371 + \left(\frac{\alpha_2}{\alpha_1} - 1\right)\mathrm{Si}(\pi\bar{y}_2)\right]\frac{N}{D} \tag{32a}$$

where

$$N = \left\{\frac{t_{01}\tau_1\theta_2}{t_{02}\tau_2\theta_1} + \left[1 - \frac{t_{01}\tau_1\theta_2}{t_{02}\tau_2\theta_1}\right]\left[\sin\pi\bar{y}_2 - \pi\bar{y}_2\cos\pi\bar{y}_2\right]\right\}$$
$$\times\left\{1 + \left[\frac{\alpha_2}{\alpha_1} - 1\right]\left[2\bar{y}_2 - \frac{\sin 2\pi\bar{y}_2}{\pi}\right]\right\} \tag{32b}$$

**This number is inferred from values for boron carbide and boron nitride, Ref. 37, pp. 23 ff and 499 ff.

and

$$D = \left\{1 + \left[\frac{\alpha_2}{\alpha_1} - 1\right]\left[2\bar{y}_2 - \frac{\sin 2\pi\bar{y}_2}{\pi}\right]\right\}^2 + \left(\frac{\omega\tau_2}{\theta_2}\right)^2\left\{\frac{\tau_1\theta_2}{\tau_2} + \left[\frac{\alpha_2}{\alpha_1}\theta_2 - \frac{\tau_1\theta_2}{\tau_2}\right]\left[2\bar{y}_2 - \frac{\sin 2\pi\bar{y}_2}{\pi}\right]\right\}^2 \tag{32c}$$

In Eqs. (32), the definitions of t_{0i}, τ_i, θ_i, and α_i are identical with those employed for a single material in the previous section. The sine-integral function Si is well-characterized and tabulated.[37]

With the best property data that could be found for boron, Fig. 3 was calculated to illustrate the effects of parameter $\bar{y}_2$ on the Debye peak. Here the depth $b = 10$ cm, but again changes in b mainly shift the peak's location on the frequency scale. The figure shows ω_{max} to be insensitive to the proportions of the beam, but the maximum ζ drops continuously from about 1.53×10^{-3} for monometallic aluminum to 7.9×10^{-4} for pure boron ($\bar{y}_2 = 0$). One concludes that this particular combination of materials holds little promise for higher damping.

Close study of Eqs. (32) suggests that very high expansivity ratios α_2/α_1 may yield greater damping than for either constituent material separately. A parameter study, therefore, was made in which all other properties were assigned their B and Al values, but α_2/α_1 was varied arbitrarily at various $\bar{y}_2$. Table 3 lists the approximate locations and heights of several Debye peaks obtained in this way. The greatest values of damping ratio are seen to exceed 10^{-2} and to occur at small $\bar{y}_1$. They are thought to be attained because of the high strain-induced temperature gradient generated in the thin aluminum layer confined between thicker slabs of a hypothetical metal with almost zero α.

Concluding Remarks

It is easy to assert that some of the results in Table 3 are quite unrealistic. Even the value $\alpha_1 = 2.9 \times 10^{-6}/°C$, quoted for the nickel-steel alloy "Invar" in the range 20-126°C by a 1930's reference,[38] yields only $(\alpha_2/\alpha_1) = 8.8$. It is known, however, that alloys can be compounded with almost zero thermal expansivity in a small neighborhood of some preassigned temperature. The author's objective in setting down these numbers is merely to emphasize the possibilities. LSS designers, it is believed, will someday find much more clever, more efficient, and more practical ways of building the needed passive damping into their structures.

Based on the discussions and analyses in this paper, some guidelines can be set down to assist the designers' task. It is emphasized that all but the first and last two of these relate primarily to monolithic structures of metallic alloy.

Without making any attempt at exhaustive coverage of the possibilities, one can list a few obvious guidelines.

1) Thermoelastic damping is affected strongly by material choice. First, this observation tends to favor reinforced plastics such as graphite/epoxy because of their high and rather frequency-independent capacity for dissipation. If a choice is to be made among metallic alloys, Table 2 shows that aluminum and magnesium are much preferable than titanium, steel, etc.

2) Several other mechanisms of anelastic relaxation represent possible damping sources in specific metallic systems. They merit further study for LSS applications.

3) The maximum possible fraction of the structure's stiffening and load-carrying capability should be assigned to flexural members of relatively thin section. For example, if classical trusswork is employed the joints should transfer bending moments in preference to pinning. This measure will, of course, also increase stiffness. It is noted, however, that it is in conflict with the principle of "fully stressed design" which is prominent in structural optimization.

4) Torsion and other deformations giving rise mainly to shear stresses are useless for damping.

5) Since the level of thermoelastic damping in metals depends linearly on the absolute temperature, the thermal balance of LSS should be designed to make T_0 as large as possible consistent with other requirements.

6) Within limits, the frequency for peak damping and the frequency range wherein damping is relatively high can be adjusted. In flexural members, the peak's location is inversely proportional to the square of the depth.

7) The first six guidelines relate to monolithic portions of the structure. Great emphasis must be placed, however, on the role of joints and interconnections. A simple illustration: If sliding ball joints of a truss are replaced by weldments, etc., as suggested in the third guideline, will the net effect be to decrease or increase the structure's overall damping?

8) Finally, it must be admitted that cases will arise where a combination of active control (at the lower end of the frequency spectrum) and artificial means such as viscoelastic inserts will prove necessary for satisfactory performance. Perhaps the value, if any, of the present paper is to point out that attractive alternatives may sometimes exist.

Acknowledgments

This research was supported in part by Air Force Office of Scientific Research under Contract AFOSR 0062 and in part by National Aeronautics and Space Administration under Grant NAG-1-97 from Langley Research Center. The author is appreciative of valuable discussions with Prof. P. Rafalski, visiting from Institute of Nuclear Research, Warsaw, Poland, and with Stanford doctoral candidate A. von Flotow. Also acknowledged is a conversation with G. Francfort whose upcoming Ph.D. dissertation[39] will contain predictions of the decay of oscillations of bars which qualitatively verify results such as those based on Eq. (25).

References

[1]Bekey, I. and Naugle, J.E., "Just Over the Horizon in Space," *Astronautics & Aeronautics*, Vol. 18, No. 5, May 1980, pp. 64–76.

[2]Hughes, P.C. and Skelton, R.E., "Controllability and Observability for Flexible Spacecraft," *Journal of Guidance and Control*, Vol. 3, Sept.-Oct. 1980, pp. 452–459.

[3]Gevarter, W.B., "Basic Relations for Control of Flexible Vehicles," *AIAA Journal*, Vol. 8, April 1970, pp. 666–672.

[4]Cannon, R.H. Jr., Private communication, Stanford University, Stanford, Calif., Jan. 1982.

[5]Wykes, J.H., "Structural Dynamic Stability Augmentation and Gust Alleviation of Flexible Aircraft," AIAA Paper 68-1067, Philadelphia, Pa., Oct. 1968.

[6]"Damping Effects in Aerospace Structures," AGARD CP-277, Oct. 1979.

[7]Bert, C.W., "Material Damping: An Introductory Review of Mathematical Models, Measures and Experimental Techniques," *Journal of Sound and Vibration*, Vol. 29, No. 2, 1973, pp. 129–153.

[8]Trudell, R.W., Curley, R.C., and Rogers, L.C., "Passive Damping in Large Precision Space Structures," *Proceedings of AIAA/ASME/ASCE/AHS 21st Structures, Structural Dynamics and Materials Conference*, Seattle, Wash., May 1980, pp. 124–136.

[9]Plunkett, R., "Friction Damping," *Damping Applications for Vibration Control*, AMD Vol. 38, edited by P.J. Torvik, ASME, N.Y., Nov. 1980, pp. 65–74.

[10]Horner, G.C., "Optimum Damper Locations for a Free-Free Beam," *Large Space Systems Technology—1980*, Vol. II, edited by F. Kopriver, NASA CP 2168, Nov. 1980, pp. 5–16.

[11]Johnson, C.D. and Kienholz, D.A., "Finite Element Prediction of Damping in Structures with Constrained Viscoelastic Layers," *Proceedings of AIAA/ASME/ASCE/AHS 22nd Structures, Structural Dynamics and Materials Conference*, Atlanta, Ga., April 1981, pp. 17-24.

[12]Holman, R.E. and Tanner, J.M., "Finite-Element Modeling Techniques for Constrained Layer Damping," *Proceedings of AIAA/ASME/ASCE/AHS 22nd Structures, Structural Dynamics and Materials Conference*, Atlanta, Ga., April 1981, pp. 8-16.

[13]Ottens, H.A., "Mathematical Formulation of Damping for Structural Response Analysis," AGARD CP-277, Oct. 1979, pp. 1-1—1-15.

[14]Bogner, F.K. and Soni, M.L., "Finite Element Vibration Analysis of Damped Structures," *Proceedings of AIAA/ASME/ASCE/AHS 22nd Structures, Structural Dynamics and Materials Conference*, Atlanta, Ga., April 1981, pp. 25-38.

[15]Brockman, R.A., "MAGNA-A Finite Element Program for the Materially and Geometrically Nonlinear Analysis of Three-Dimensional Structures Subjected to Static and Transient Loading," AFWAL-TR-80-3152, Jan. 1981.

[16]Crandall, S.H., "On Scaling Laws for Material Damping," NASA TN D1467, Dec. 1962.

[17]Torvik, P.J., ed., *Damping Applications for Vibration Control*, AMD Vol. 38, ASME, N.Y., Nov. 1980.

[18]Lazan, B.J., *Damping of Materials and Members in Structural Mechanics*, Pergamon Press, Oxford, 1960.

[19]Bert, C.W., "Composite Materials: A Survey of the Damping Capacity of Fiber Reinforced Composites," *Damping Applications for Vibration Control*, AMD Vol. 38, edited by P.J. Torvik, ASME, N.Y., Nov. 1980, pp. 53-63.

[20]Nowick, A.S. and Berry, B.S., *Anelastic Relaxation in Crystalline Solids*, Academic Press, New York, 1972.

[21]Nowacki, W., *Thermoelasticity*, Addison-Wesley Publishing Co., Reading, Mass., Pergamon Press, London, and PWN, Warsaw, 1962.

[22]Zener, C., "Internal Friction in Solids, Part I," *Physical Review*, Vol. 52, Aug. 1, 1937, pp. 230–235.

[23]Zener, C., "Internal Friction in Solids, Part II," *Physical Review*, Vol. 53, Jan. 1, 1938, pp. 90–99.

[24]Zener, C., *Elasticity and Anelasticity of Metals*, University of Chicago Press, Chicago, Ill., 1948.

[25]Bennewitz, K. and Rötger, H., "On the Internal Friction of Solid Bodies: Absorption Frequencies of Metals in the Acoustic Range," (in German), *Physikalische Zeitschrift*, Vol. 37, 1936, pp. 578-588.

[26]Biot, M.A., "Thermoelasticity and Irreversible Thermodynamics," *Journal of Applied Physics*, Vol. 27, No. 3, March 1956, pp. 240-253.

[27]Tasi, J., "Thermoelastic Dissipation in Vibrating Plates," *Transactions of the ASME, Journal of Applied Mechanics*, Vol. 30, No. 4, Dec. 1963, pp. 562-567.

[28]Tasi, J. and Herrmann, G., "Thermoelastic Dissipation in High-Frequency Vibrations of Crystal Plates," *Journal of the Acoustical Society of America*, Vol. 36, No. 1, Jan. 1964, pp. 100-110.

[29]Bisplinghoff, R.L. and Ashley, H., *Principles of Aeroelasticity*, John Wiley, New York, 1962 (reprinted by Dover Publications, New York).

[30]Nowinski, J.L., *Theory of Thermoelasticity with Applications*, Sijthoff and Noordhoff, the Netherlands, 1978.

[31]Eshbach, O.W. and Souders, M., eds., *Handbook of Engineering Fundamentals*, 3rd Ed., John Wiley & Sons, New York, 1975, Tables on pp. 492 ff., 1324-5 and 1332-3.

[32]Reynolds, W.C. and Perkins, H.C., *Engineering Thermodynamics*, McGraw Hill Book Co., New York, 1977, see especially p. 659.

[33]Vorlicek, P.J., "Material Damping of Aluminum and Graphite/Epoxy in a Simulated Zero-Gravity Environment," M.I.T. Space Systems Lab., Cambridge, Mass., Rept. 13-81, Jan. 1981.

[34]Van Dyke, M.D., *Perturbation Methods in Fluid Mechanics*, Academic Press, New York, 1964.

[35]Meirovitch, L., *Methods of Analytical Dynamics*, McGraw-Hill Book Co., New York, 1970, pp. 222-224.

[36]Bagley, R.L. and Torvik, P.J., "Fractional Calculus—A Different Approach to the Finite Element Analysis of Viscoelastically Damped Structures," *Proceedings of AIAA/ASME/ASCE/AHS 21st Structures, Structural Dynamics and Materials Conference*, Seattle, Wash., May 1980, pp. 1-7.

[36]Touloukian, Y.S., ed., *Thermophysical Properties of High Temperature Solid Materials*, Vol. 5, The Macmillan Co., New York, 1967.

[37]Abramowitz, M. and Stegun, I.A., eds., *Handbook of Mathematical Functions*, N.B.S. Applied Mathematics Series No. 55, 1972, pp. 231-232 and Table 5.1.

[38]Mochel, N.L., "Thermal Expansion of Metals," *Symposium on Effects of Temperature on the Properties of Metals*, edited by H.W. Gillett, ASTM, Philadelphia, Pa., and ASME, New York, Dec. 1931, Table VI, p. 705.

[39]Francfort, G.A., "Two Variational Problems in Thermoelasticity," Seminar talk given Feb. 11, 1982, and based on research for the Ph.D. degree, Division of Applied Mechanics, Stanford University, Stanford, Calif., 1982.